Abstract

The core idea of this article is nested parametrization in the context of isogeometric analysis. The method has been inspired by trimming procedures and can be applied to different applications like local modifications and enhancements of thin-walled structures or coupling of two overlying elements by embedding one in the other.

A remodeling for an explicit representation of the boundaries is avoided, which would be contradictory to the aim of isogeometric analysis of using the original CAD model. The nested entity is directly linked to the super element, however its element formulation is independent of the super element formulation within this article. The derivation, implementation and application is shown in particular for one-dimensional entities embedded into 2D domains. Consequently, the definition of an embedded curve in the surface is required and realized by using NURBS curves in the parameter space of the corresponding surface. This curve with its respective predefined base vectors serves as the basis for new element formulations or adaptation of already developed element formulations, which are based on the geometric description of a curve.

In detail, an adapted formulation of the recently developed nonlinear isogeometric spatial Bernoulli Beam by the authors is presented in this paper. Furthermore, those embedded curves are used for line supports and loads, as well as a mass manipulation. The accuracy of the proposed element formulation is verified by several benchmark examples and the potential for future applications is briefly revealed.

Keywords: Nonlinear isogeometric analysis, Nested parameterization, Embedded beam element, Trimming, NURBS

1. Introduction

In the field of computational simulation, one can find a large number of applications where a huge super domain is influenced by local effects. Only to name a few, this can be a beam supported membrane, a rib-enforced metal sheet or introducing thermal energy on a line by means of a built-in heat-source. All these problems have to be solved as one single physical system comprising all sub-members. Considering the local quantities can entail much manual effort for subdividing, remodeling and meshing, which is contradictory to the aim of isogeometric analysis of processing the original CAD geometry. A matching grid is no longer required and additional mapping matrices have not to be established. Therefore, the embedding becomes independent of the discretization of the super domain. Furthermore, complex coupling formulations for linking local property to the super domain can be avoided.

The concept of nested parametrization maps the local quantities into the parameter space of the super domain, which is inspired by trimming curves in the IBRA approach of Breitenberger et al. [1]. This implies that the local entities are fully linked to the super domain and the same control points and respective degrees of freedom determine its shape in the geometry space. We restrict the problem definition in this publication without loss of generality to a 1D element in a 2D domain, thus a curve embedded into a surface. Computer aided design (CAD) already provides functions in order to construct those embedded curves. Furthermore, the basis functions – non-uniform rational B-Splines (NURBS) [2, 3] – are very suitable for nesting and application in the finite element analysis (FEA).
The application in simulation is called *isogeometric analysis* (IGA) [4, 5]. Due to several advantages like higher smoothness and direct use of geometry description, many element formulations have been developed. In the case of structural analysis application, these are e.g. for a solid [4, 6] and for shells and membranes [7, 8, 9, 10, 11, 12, 13, 14, 15], among others. In parallel, one dimensional element formulations, such as cables and rods, can be found in [16, 17, 18, 19]. The proposed concept provides a possibility in order to combine or manipulate them. Surface independent element descriptions, which are derived from the continuum can be seen as a template and be adapted for embedding. A similar application of this approach was done by Philipp *et al.* [15] with embedding an edge cable into a membrane on trimming edges.

In this paper, the recently developed nonlinear isogeometric spatial Bernoulli beam by the authors [19] is adapted in order to obtain an embedded beam element formulation. This beam can be attached rigidly to the surface or by a hinge, *i.e.* only the translational displacements of the center line is coupled. Both cases are exemplified in the following. The embedded line enables also to apply further properties such as line loads or supports to a surface without being limited to edges of the surface.

The outline of this work is as follows: Section 2 presents the nesting of NURBS geometries after a brief introduction to NURBS. Essential variations for NURBS-based embedded FE-elements are shown. Basic formulas for the application in structural mechanics can be found in Section 3. The transfer of the concept of nested parametrization to structural element formulations is the main subject of Section 3. The implementation of these element formulations is verified in the application within several selected benchmarks. Furthermore, some examples for potential applications have been simulated. These results are presented in Section 4. Section 5 draws the conclusion and describes future topics and possibilities for extensions.

### 2. Nested parametrization in isogeometric analysis

Isogeometric analysis (IGA) is an approach to integrate analysis (FEA) into the design environment (CAD). This is realized by using the same basis functions for both parts. Hence, a considerable amount of time can be saved by omitting the conversion between the two models [4]. These basis functions present a perfect basis for implicit element formulations, since NURBS-patches provide high accuracy in the solution field combined with a parameter space over a huge domain.

#### 2.1. Non-uniform rational B-Splines – NURBS

Non-Uniform Rational B-Splines (NURBS) are widely used as basis functions for isogeometric analysis, since they are one of the basic geometry descriptions in CAD. The continuous entity is described by discrete control points $P$, which are generally non-interpolating. The NURBS geometry $C(\xi)$ or $S(\xi, \eta)$ is defined as the sum over all control points $P$ multiplied with their respective NURBS basis function $R$.

\[
C(\xi) = \sum_{i=1}^{n} R_{i,p}(\xi)P_i \quad \quad S(\xi, \eta) = \sum_{i=1}^{n} \sum_{j=1}^{m} R_{i,j,pq}(\xi, \eta)P_{ij},
\]

The NURBS description is a weighted B-Spline description with a polynomial degree $p$ and $q$ in the 2D case.

\[
R_{i,p}(\xi) = \frac{N_{i,p}(\xi)w_i}{\sum_{j=1}^{n} N_{j,p}(\xi)w_j} \quad \quad R_{i,j,pq}(\xi, \eta) = \frac{N_{i,p}(\xi)M_{j,q}(\eta)w_{ij}}{\sum_{k=1}^{n} \sum_{l=1}^{m} N_{k,p}(\xi)M_{l,q}(\eta)w_{kl}}
\]

The construction of the B-Spline functions $N_{i,p}$ and $M_{j,q}$ (see Fig. 1) with the Cox–deBoor recursion formula and further details on NURBS can be found in [2].

#### 2.2. Embedded geometries in isogeometric elements

The NURBS parameter space provides a suitable domain for embedding. This domain is called super domain in the following. The idea is to define local quantities, which are related to a super domain, within this super domain, *i.e.* embed them into the description of the super domain. The local enhancements are then expressed by the degrees
of freedom of the super domain and supplementary coupling can be avoided. The dimensionality of the embedded domain is typically smaller. In analogy to trimming, the embedded object is defined by another NURBS geometry in the parameter space. The sub domain is thus automatically coupled to the super domain. This is exemplified for a curve embedded into a surface. Quantities with respect to the parameter space of the surface will be denoted with $\bar{\text{(\textbullet)}}$ in the following. The control points $\bar{\mathbf{P}}_i$ then have the coordinates of the parameter space $(\theta_1, \theta_2)$ of the surface. The computation of the NURBS curve in the parameter space is done analogously to Eq. (1).

$$\bar{\mathbf{C}}(\bar{\theta}^1) = \left[ \begin{array}{c} \theta^1(\bar{\theta}^1) \\ \theta^2(\bar{\theta}^1) \end{array} \right] = \sum_{i=1}^{n} R_{i,\theta}^p(\bar{\theta}^1) \bar{\mathbf{P}}_i$$

(3)

This kind of description is directly available from CAD, when constructed by intersection of a surface and a curve or other surface (see Fig. 2). The curve description with knot vector and control points in the parameter space results from a surface-to-surface intersection (SSI) problem [20, 21, 22, 23]. In general, it is not possible to describe...
the resulting curve by the same knot vector nor to exactly represent the original (intersecting) curve. The precision is
dependent on the tolerances of the CAD-system respectively the SSI-algorithm. Note that the position of those control
points within the parameter space of the super element does not change during the analysis. The deformation of the
curve is only implied by the deformation of the surface (cf. Fig. 3).

![Figure 3: Embedded curve in the (a) undeformed surface and (b) in the deformed surface. The surface knot vector is \( \Xi_{srf} = H_{srf} = [0, 0, 0, 1, 1, 1] \) and the curve knot vector is equal to \( \Xi_{crv} = [0, 0, 1, 1] \) for a linear curve. The control points of the curve are located at \( \vec{P}_1 = [0, 0]^T \) and \( \vec{P}_2 = [1, 1]^T \). Note that the curve in the deformed configuration is still represented by a linear curve in the parameter space.](image)

The curve in the geometric space can then be expressed as follows (see Fig. 4):

\[
C(\vec{\theta}_1) = \sum_{i=1}^{n} \sum_{j=1}^{m} R_{i,j,pq}(\vec{\theta}_1) P_{ij} = \sum_{i=1}^{n} \sum_{j=1}^{m} R_{i,j,pq}(\vec{\theta}_1, \vec{\theta}_2(\vec{\theta}_1)) P_{ij} = \sum_{i=1}^{n} \sum_{j=1}^{m} R_{i,j,pq} \left( \sum_{k=1}^{n} R_{k,i,p} (\vec{\theta}_1) \right) P_{ij}
\]

(4)

Table 1 gives an overview of all aforementioned curve definition and their mapping space.

<table>
<thead>
<tr>
<th>Curve Definition</th>
<th>Mapping Space</th>
</tr>
</thead>
<tbody>
<tr>
<td>General curve in geometry space</td>
<td>( C : \vec{\theta}_1 \rightarrow \mathbb{R}^3 )</td>
</tr>
<tr>
<td>Embedded curve in parameter space</td>
<td>( \overline{C} : \vec{\theta}_1 \rightarrow \mathbb{R}^2 )</td>
</tr>
<tr>
<td>Embedded curve in geometry space</td>
<td>( C : \vec{\theta}_1 \rightarrow \mathbb{R}^3 )</td>
</tr>
</tbody>
</table>

Table 1: Overview over curve definitions with respective mapping space.

Additionally, a local coordinate system \( \mathbf{B}_i \) is introduced in order to describe a local 3D continuum. The base
vectors \( \mathbf{B}_i \) are defined as unit vectors and orthogonal respectively tangential to the curve (see also Fig. 4).

\[
\mathbf{B}_i = \frac{\vec{B}_i}{||\vec{B}_i||_2}
\]

(5)

\( \vec{B}_i \) denotes the not normalized base vectors. These base vectors of the curve can be derived from the base vectors
of the surface \( A_i \). The first base vector is aligned to the tangent of the curve, the second corresponds to the surface

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The base vector \( \hat{B}_3 \) can be rewritten with the help of a pseudo parameter \( \bar{\theta} \) and the triple product expansion [1]:

\[
\hat{B}_3 = A_1 \frac{\partial X_i}{\partial \bar{\theta}^1} + A_2 \frac{\partial X_i}{\partial \bar{\theta}^2}
\]

with \( \frac{\partial X_i}{\partial \bar{\theta}^1} = \hat{B}_1 \cdot A_2 \) and \( \frac{\partial X_i}{\partial \bar{\theta}^2} = -\hat{B}_1 \cdot A_1 \) (9)

2.3. NURBS-based embedded finite elements

Variations with respect to a discrete global vector of degrees of freedom \( \hat{u} \) of the geometric quantities are usually required for the formulation of a finite element. In the following, discrete nodal values will be denoted by \( (\bullet) \). Each node has the degrees of freedom (DOFs) of the element formulation of the surface element. These are e.g. three displacement DOFs \( u, v, w \). Optionally additional DOFs for the embedded element, like a rotation around the center line \( \phi \), can be applied. Note that these additional DOFs are also attributed to the surface control points even though they do not influence the surface element itself.

\[
\hat{u} = \begin{bmatrix} \hat{u}^{1,1} & \hat{\phi}^{1,1} & \hat{\gamma}^{1,1} & (\hat{\phi}^{1,1}) & \ldots & \hat{u}^{n,m} & \hat{\phi}^{n,m} & \hat{\gamma}^{n,m} & (\hat{\phi}^{n,m}) \end{bmatrix}^T
\]

The center line \( \mathbf{x}_C \) of the deformed configuration can be described as a function of the initial coordinates of the surface control points and their respective displacements.

\[
\mathbf{x}_C(\bar{\theta}^i) = \sum_{i=1}^n \sum_{j=1}^m R_{ij,pq} \left( \bar{\theta}^1(\bar{\theta}^i), \bar{\theta}^2(\bar{\theta}^i) \right) \hat{x}^{ij} = \sum_{i=1}^n \sum_{j=1}^m R_{ij,pq} \left( \sum_{i=1}^n \sum_{j=1}^m R_{ir,p} \left( \bar{\theta}^r \right) \hat{P}_r \right) \cdot (\hat{x}^{ij} + \hat{u}^{ij})
\]

Figure 4: Surface with embedded curve and the underlying parameter space with the defining NURBS curve: Definition of the base vectors of the embedded curve \( B_i \) and of the base vectors of the underlying surface \( A_i \).
Since all parameters of the undeformed geometry are invariant to the variation, the variation $x_{C,r}$ of the position vector with respect to the variation parameter $r$ (the $r$-th component of $\hat{\mathbf{u}}$ as defined in Eq. (10)) can be written as:

$$x_{C,r} = \sum_i \sum_j R_{ij,pq} (\hat{X}_{ij} + \hat{u}_{ij})_r = \sum_i \sum_j R_{ij,pq} \hat{u}_{ij,r} \quad \text{with } r \in [0, i \cdot j \cdot n_{DOFs,CP} - 1] \quad (12)$$

The description of the center line also influences the variation of the base vectors $\hat{b}_i$:

$$\hat{b}_{1,r} = x_{C,1,r} = \sum_i \sum_j R_{ij,pq} \hat{u}_{ij} = \sum_i \sum_j \left( \frac{\partial R_{ij,pq}}{\partial \theta_1} \hat{u}_{ij} \right) + \left( \frac{\partial R_{ij,pq}}{\partial \theta_2} \hat{u}_{ij} \right) \quad (13)$$

$$\hat{b}_{2,r} = (x_1 \times x_2)_r = \sum_i \sum_j \left( \frac{\partial R_{ij,pq}}{\partial \theta_1} \hat{u}_{ij} \right) \times \left( \frac{\partial R_{ij,pq}}{\partial \theta_2} \hat{u}_{ij} \right) + \left( \frac{\partial R_{ij,pq}}{\partial \theta_1} \hat{u}_{ij} \right) \times \left( \frac{\partial R_{ij,pq}}{\partial \theta_2} \hat{u}_{ij} \right) \quad (14)$$

The integration of the terms related to the curve element for FE analysis is required. As commonly applied this is realized as a Gauss-quadrature. Note that the coordinates and weights of the Gauss points are derived from the elements of the curve in the parameter space of the surface. The minimal number of Gauss points is derived by the largest polynomial degree $p + 1$ either of the parameter curve or the surface. The refinement of NURBS curves in the parameter space can be executed in the same manner as for space curves. This refinement does not alter the geometry. Only the number of control points in the parameter space is increased, but since no additional DOFs are introduced, it does not enrich the solution space. Nevertheless, a refinement of the curve in the range of the refinement of the surface is beneficial, since they share the same DOFs. This can be realized by, e.g., inserting knots at every crossing of the parameter curve over a knot line of the surface (see Fig. 5). The number of elements is then similar and the numerical error of the integration is in the same order [1].

![Figure 5](image_url)

**Figure 5**: Embedded curve ($p_{crv} = 2$) with Gauss points (green squares) and respective knots (blue lines) for (a) unrefined curve and (b) refined curve with knot insertion at the intersections with the knot lines of the surface and order elevation corresponding to the polynomial degree of the surface $p_{srf} = 3$.

### 3. Exemplary derivation of embedded structural elements

#### 3.1. Structural mechanics

The embedded curves as defined in Section 2 can be used in order to apply mechanically motivated enhancements to thin-walled structures. The considered problems are composed of two types of elements of different dimensions,
i.e. two domains. There is the two-dimensional domain $\Omega^S$ for the surface elements. The one-dimensional elements are defined on the domain $\Omega^C$, which is a subset of $\Omega$. Note that the embedded curve is not restricted to the trimmed surface domain $\Omega^S$, but to the untrimmed surface domain $\Omega^s$. The boundaries consist of the respective Dirichlet boundaries $\Gamma_D$ and the Neumann boundaries $\Gamma_N$ (cf. Fig. 6).

The Principle of Virtual Work is used to derive the element formulations in the sequel. The inner and outer forces of surface and curve are multiplied by a respective virtual displacement $\delta u$. If the sum of the performed internal and external work is zero, the system is in equilibrium [24]. It can be derived as follows for an independently defined, but coupled system:

$$\delta W = -\delta W_{int}^S + \delta W_{ext}^S - \delta W_{int}^C + \delta W_{ext}^C + \delta W_{coupling} = 0 \quad (15)$$

The virtual work is divided in terms resulting from the surface ($\delta W_{int}^S$, $\delta W_{ext}^S$) and curve elements ($\delta W_{int}^C$, $\delta W_{ext}^C$). An additional term for the coupling ($\delta W_{coupling}$) is added in the classic approach in order to let them interact for non matching grids, where degrees of freedom (DOFs) cannot be shared. The novel approach has the aim to describe those two domains with one formulation, i.e. to express one domain within the other. Consequently, the coupling term $\delta W_{coupling}$ is not necessary anymore. The “monolithic” formulation of the virtual work uses only the basis functions of the surface in order to create test functions. Embedding the curve into the surface enables strong coupling, i.e. the displacements of surface and curve are inherently consistent. Therefore, Eq. (15) can be rewritten as:

$$\delta W = -\int_{\Omega^S} S^S : \delta E^S \, dx - \int_{\Omega^C} S^C : \delta E^C \, dx$$

$$+ \left( \int_{\Gamma_{NS}^S} t^S : \delta u^S \, ds + \int_{\Omega^S} \rho^0 B^S : \delta u^S \, dx + \int_{\Gamma_{NS}^C} t^C : \delta u^C \, ds + \int_{\Omega^C} \rho^0 B^C : \delta u^C \, dx \right) = 0 \quad (16)$$

where $\Omega$ describes the domain and $\partial \Omega$ the boundary in the undeformed state. The second Piola-Kirchhoff (PK2) stresses are denoted as $S$ and the energetically conjugated virtual Green-Lagrange (GL) strains caused by the virtual displacements $\delta u$ are denoted as $\delta E$. The external forces consist of body forces $B$ and boundary forces $t$. The material density is named $\rho_0$.

The Finite Element Method is used to solve the system. Hence, a discretization is necessary. The components $R_r$ of the residual force vector are defined by the variation of Eq. (16) with respect to the discretization variables $\delta u_r$.

$$\delta W = \sum \frac{\partial W}{\partial u_r} \delta u_r = \sum R_r \delta u_r = 0 \quad (17)$$

This yields for arbitrary variations $\delta u_r$:

$$R_r = \frac{\partial W}{\partial u_r} = 0 \quad (18)$$
A linearization at the current displacement state $u^*$ becomes necessary, if an iterative method like the Newton-Raphson method is used for the solution of the mechanical system. Therefore, the displacement increments $\Delta u_i$ and the components $K_{rs}$ of the tangential stiffness matrix are introduced:

$$LIN(R_i) = \frac{\partial W}{\partial u_i} + \frac{\partial^2 W}{\partial u_i \partial u_i} \Delta u_i = R_i |_{u^*} + \sum_s \frac{\partial R_s}{\partial u_i} \Delta u_i = 0$$ \hspace{1cm} (19)

As a conclusion, the residual force vector and stiffness matrix can be decomposed into contributions of the internal and external forces: The residual force vector and stiffness matrix are partitioned into terms caused by internal respectively external forces

$$R_r = \frac{\partial W}{\partial u_i} = \frac{\partial W_{\text{int}}}{\partial u_i} + \frac{\partial W_{\text{ext}}}{\partial u_i} = -\left(F_{\text{int}}^r + F_{\text{ext}}^r\right)$$ \hspace{1cm} (20)

$$K_{rs} = \frac{\partial R_r}{\partial u_i} = \frac{\partial^2 W}{\partial u_i \partial u_i} = \frac{\partial^2 W_{\text{int}}}{\partial u_i \partial u_i} + \frac{\partial^2 W_{\text{ext}}}{\partial u_i \partial u_i} = K_{rs}^{\text{int}} + K_{rs}^{\text{ext}}$$ \hspace{1cm} (21)

The load stiffness terms $K_{rs}^{\text{ext}}$ are neglected within this contribution. For more details, refer to e.g. Schweizerhof and Ramm [25].

3.2. Beam element

The shown structural enhancement for the embedded curve is a beam element. There are two convenient attachments for a beam to a surface, which is described fixedly in the parameter space: (i) rigidly attached and (ii) hinge-attached.

How to apply the nested parametrization concept to the element formulation of the nonlinear spatial isogeometric Bernoulli beam (BB) by Bauer et al. [19] for those two cases is shown in the following. The BB formulation provides Bernoulli kinematics with warping-free torsion. It is an element formulation based on a geometric description of the continuum. Every point of the continuum of the beam can be addressed by the center line and the corresponding base vectors.

Analogously to [19], $X$ respectively $x$ are the position vectors for the continuum of the beam and are defined as follows (see also Fig. 7):

$$X \left(\theta^1, \theta^2, \theta^3\right) = X_C \left(\theta^1\right) + \hat{\theta}^2 B^B_2 \left(\theta^1\right) + \hat{\theta}^3 B^B_3 \left(\theta^1\right)$$ \hspace{1cm} (22a)

$$x \left(\theta^1, \theta^2, \theta^3\right) = x_C \left(\theta^1\right) + \hat{\theta}^2 b^B_2 \left(\theta^1\right) + \hat{\theta}^3 b^B_3 \left(\theta^1\right)$$ \hspace{1cm} (22b)

The center line $X_C$ (respectively $x_C$) is defined by the curve in the parameter space. The remaining input parameters, i.e. the base vectors $B^B_2$ respectively $b^B_2$ can be described by transferring the original element formulation to the nested curve properties. Note that the surface normal $A_3$ provides a unique reference. Hence, the moving trihedral with respective $A$-operator, which has been introduced in [19] in order to create a reference trihedral for the description of the cross section, is not needed in the embedded formulation. The kinematic relations will be presented in Section 3.2.3.

3.2.1. Rigidly attached beam element

Rigidly attached means that the cross section of the beam is fully linked to the surface. In other words when the surface is twisted, also the beam is twisted. This corresponds e.g. to ribs on a shell structure.

Consequently, the local coordinate system $B_i$ as introduced in Section 2.2, which is strictly derived from the surface, can be applied as the base vectors $B^B_i$ of the continuum of the beam (Fig. 7). The not normalized base vector $\tilde{B}_1$ is used in tangential direction in order to represent longitudinal changes. In contrast to the original element formulation from [19], the fourth DOF $\psi$ for torsion is not applicable, since the twisting is tracked solely by the surface.
3.2.2. Hinge-attached beam element

An application for this formulation is e.g. a beam supported membrane. In this case, the orientation of the cross section of the beam is independent of the surface. Only the center line stays attached. The base vectors $B^\alpha$ and $b^\alpha$ with $\alpha \in \{2, 3\}$ are derived by modifying the reference base vectors $B_\alpha$ respectively $b_\alpha$ from the surface as defined in Section 2.2. Therefore, a fourth, torsional DOF is activated. It measures the relative rotation around the curve tangent. The reference vectors are then rotated by a correction angle in the reference and/or the torsional DOF in the deformed configuration (cf. Fig. 8).

\[
B^\alpha(\bar{\theta}^i) = R_{B_\alpha}(\Psi) B_\alpha(\bar{\theta}^i) \tag{23a}
\]
\[
b^\alpha(\bar{\theta}^i) = R_{b_\alpha}(\psi) R_{b_\alpha}(\Psi) b_\alpha(\bar{\theta}^i) \tag{23b}
\]

The angle is applied with a Euler–Rodrigues rotation matrix [26] with $B_1$ respectively $b_1$ as axis of rotation. For more details within the context of beam elements see [19].

\[
R_{b_1}(\psi) = I \cos(\psi) + \sin(\psi) \ b_1 \times I + \left(1 - \cos(\psi)\right) b_1 \otimes b_1 = I \cos(\psi) + \sin(\psi) \ b_1 \times I \tag{24}
\]
3.2.3. Nonlinear kinematics

The kinematics are derived directly from the geometric description in combination with the laws of continuum mechanics. Strains and stresses are determined by using the Green-Lagrange (GL) strain tensor and the energetically conjugated second Piola-Kirchhoff (PK2) stress tensor. More details and explanations can be found in [19].

The base vectors of the continuum of the curve element are defined as:

\[ \mathbf{G}_i = \frac{\partial \mathbf{X}}{\partial \bar{\theta}^i} = \mathbf{X}_j, \quad \mathbf{g}_i = \frac{\partial \mathbf{x}}{\partial \bar{\theta}^i} = \mathbf{x}_j \]

This results in the following with Eq. (22a) resp. Eq. (22b).

\[ \mathbf{G}_1 = \frac{\partial \mathbf{X}}{\partial \bar{\theta}^1} = \mathbf{B}_1 + \bar{\theta} \mathbf{B}_{2,1}^B + \bar{\theta}^2 \mathbf{B}_{3,1}^B, \quad \mathbf{g}_1 = \frac{\partial \mathbf{x}}{\partial \bar{\theta}^1} = \mathbf{b}_1 + \bar{\theta} \mathbf{b}_{2,1}^B + \bar{\theta}^2 \mathbf{b}_{3,1}^B \]  \hspace{1cm} (26a)

\[ \mathbf{G}_2 = \frac{\partial \mathbf{X}}{\partial \bar{\theta}^2} = \mathbf{B}_2^B, \quad \mathbf{g}_2 = \frac{\partial \mathbf{x}}{\partial \bar{\theta}^2} = \mathbf{b}_2^B \]  \hspace{1cm} (26b)

\[ \mathbf{G}_3 = \frac{\partial \mathbf{X}}{\partial \bar{\theta}^3} = \mathbf{B}_3^B, \quad \mathbf{g}_3 = \frac{\partial \mathbf{x}}{\partial \bar{\theta}^3} = \mathbf{b}_3^B \]  \hspace{1cm} (26c)
Then, the GL strain tensor can be defined as follows:

\[ E_{ij} = \frac{1}{2} (g_{ij} - G_{ij}) \]

where \( i, j \in \{1, 2, 3\} \) and \( G_{ij} = G_i \cdot G_j \) (analogously: \( g_{ij} \))

The respective strains for the beam formulation are consequently computed in the following way.

\[ E_{11} = \frac{1}{2} \left( \left( b_{11} \cdot \hat{b}_1 - \hat{b}_1 \cdot \hat{b}_1 \right) + \theta^\alpha \left( b_{21} \cdot \hat{b}_2 - \hat{b}_2 \cdot \hat{b}_2 \right) + \theta^\beta \left( b_{31} \cdot \hat{b}_3 - \hat{b}_3 \cdot \hat{b}_3 \right) \right) \]

\[ E_{1\alpha} = \frac{1}{2} \theta^\alpha \left( b_{\alpha 1} \cdot \hat{b}_\alpha - \hat{b}_\alpha \cdot \hat{b}_\alpha \right) \]

with \( (\alpha, \beta) \in \{(2, 3), (3, 2)\} \)

Note that square order terms are neglected for \( E_{11} \) due to the assumed slenderness of the beam. All other, not-presented strains are equal to zero. The strain related to the normal force is denoted by \( \epsilon \), the strains corresponding to bending and torsion by \( \kappa_{21} / \kappa_{31} \) respectively \( \kappa_{23} / \kappa_{32} \).

Following the derivation of the element formulation in [19], the following weak form of the equilibrium evolves for both attachments with \( B_a \) being principal axes of the cross section:

\[ \delta W_{\text{int}} = - \int_{\Omega \subset \mathbb{R}^3} S : \delta E \, d\Omega \]

\[ = - \int_{L} \frac{E}{||B||^4} \cdot (A \epsilon \delta \epsilon + I_B, \kappa_{21} \delta \kappa_{21} + I_B, \kappa_{31} \delta \kappa_{31}) + \frac{GI_B}{||B||^2} \left( \frac{1}{2} \kappa_{32} \delta \kappa_{32} + \frac{1}{2} \kappa_{23} \delta \kappa_{23} \right) \, dL, \]

where \( A \) is the area of the cross section and \( I_B \) the moments of inertia with respect to the base vectors.

### 3.3 Mass attributed curve elements

The developed concept to embed curve-shaped mechanical properties into isogeometric analysis of thin-walled structures can be beneficially applied for a variety of further applications. Following the idea of beam enforcements, the NURBS-embedded curve concept can also be used to add additional mass which is distributed along a curve on the free-form structure. These nonstructural masses (i.e., inertia without stiffening effects) can influence e.g., the eigenfrequencies and mode shapes. It is derived from the dynamic part of the virtual work, which can be added to Eq. (16).

\[ \delta W_{\text{dyn}} = - \int_{\Omega} \rho \cdot \delta \mathbf{u} \, d\Omega = - \int_{\Omega} \rho \left( \sum_k \sum_i R_{(kl,pq)} \hat{u}^{ij} \right) \left( \sum_j \sum_i R_{(ij,pq)} \hat{u}^{ij} \right) d\Omega \]

\[ = - \sum_k \sum_i \sum_j \sum_p \sum_q \hat{u}^{ij} R_{(kl,pq)} R_{(ij,pq)} d\Omega \delta \mathbf{u}^{ij} = - \sum_k \sum_i \sum_j \sum_p \sum_q \hat{u}^{ij} M_{ijkl} \delta \mathbf{u}^{ij} \]

Together with the preintegrated integral over the domain for dimensionally reduced elements

\[ \int_{\Omega} \rho \cdot (...) d\Omega = \rho \cdot A \int_{L} (...) dL = \rho_0 \int_{L} (...) dL, \]

the variation within the dynamic work contribution results in the mass matrix:

\[ M_{ij} = \begin{cases} \rho_0 \cdot \int_{L} R_{kl,pq} R_{ij,pq} dL & r^\% n_{\text{DOF}/\text{CP}} = s^\% n_{\text{DOF}/\text{CP}} \\ 0 & \text{otherwise} \end{cases} \]

where \( i, j, k \) and \( l \) are the numbers of the control point corresponding to the \( r \)-th respectively \( s \)-th component of the discretized displacement vector \( \mathbf{u} \). Note that \% denotes the modulo-operation and \( n_{\text{DOF}/\text{CP}} \) the number of DOFs per control point, which indicates that only DOFs in the same direction influence each other.
3.4. Boundary conditions

Introducing boundary conditions on such embedded curves is a further obvious application, which in classic FE analysis can cause a big amount of work during the set-up of the simulation. An application example is shown in the following in Section 4.2.1.

3.4.1. Dirichlet boundary conditions

Also linear boundary conditions, such as supports can be applied to the surface at any arbitrary position. A penalty approach is chosen and the coupling solution within IBRA presented by Breitenberger et al. [1] can serve as template for this context. Hence, an additional term is added to the equilibrium condition.

\[ \delta W = -\delta W_{\text{int}} + \delta W_{\text{ext}} + \delta W_{\text{BC B-Rep}} \]  \hfill (34)

This term may be written as follows for restraining respectively prescribing the displacement \( \mathbf{u} \) and rotation around the curve’s tangent \( \omega_{\mathbf{B}_1} \):

\[ \delta W_{\text{BC disp B-Rep}} = -\alpha_{\text{disp}} \int_{\Gamma_c} (\mathbf{u} + \mathbf{u}_0) \cdot \delta (\mathbf{u} + \mathbf{u}_0) \, d\Gamma_c \]  \hfill (35)

\[ \delta W_{\text{BC rot B-Rep}} = -\alpha_{\text{rot}} \int_{\Gamma_c} (\omega_{\mathbf{B}_1} + \psi) \cdot \delta (\omega_{\mathbf{B}_1} + \psi) \, d\Gamma_c \]  \hfill (36)

In contrast to the B-Rep coupling formulation in [1], there is no mutual coupling between two patches: In fact, the patch is coupled to a fixed displacement \( \mathbf{u}_0 \) or rotation \( \omega_{\mathbf{B}_1} \) of e.g. \( 0 \) along an arbitrary curve on the surface. The rotation axis is \( \mathbf{B}_1 \) and the rotation is tracked by \( \mathbf{B}_2 \).

\[ \omega_{\mathbf{B}_1} = \arcsin ( (\mathbf{B}_2 \times (\mathbf{b}_2 - \mathbf{B}_2)) \cdot \mathbf{B}_1 ) \]  \hfill (37)

3.4.2. Neumann boundary conditions

Neumann boundary conditions such as line loads cannot directly be applied on the control points. An integration along the line has to be performed for the mapping on respective control points and therefore consistent application to the right hand side. The components of the consistent nodal force vector \( \hat{F}_{ij} \) for a load \( q(\hat{\theta}^l) \) are computed as follows:

\[ \hat{F}_{ij} = \int_{\Gamma_c} R_{ijpq}(\hat{\theta}^l) q(\hat{\theta}^l) \, d\Gamma_c \]  \hfill (38)

4. Numerical examples

4.1. Verification of element formulations and implementation

In this section some selected examples are presented. They verify the nonlinear element formulation and the correct implementation of the presented spatial Plug-On Beam (POB) in Section 3.2. Therefore, different exemplary loading scenarios and geometries are tested.

4.1.1. Bending–torsion interaction

The bending–torsion interaction is tested with the quadrant. It is an initially plane Kirchhoff-Love shell as proposed in [7]. The beam is plugged onto the outer edge. The dimensions of the structural quantities are visualized in Fig. 9 for better illustration of the set-up. Within the CAD environment only the mid surface of the shell is visualized. The quadrant is hinge supported at one side. Only the beam is clamped (also rotationally). The beam rotation is independent of the surface, i.e. hinge-attached. A line load \( p_z \) in z-direction is applied on the free end of the quadrant.
The refinement and polynomial degree in tangential direction are variable. The radial refinement and polynomial degree are constant and set to three elements and $p = 2$. The displacement of the tip of the beam and the inner forces inside the beam at point $A$ are depicted in Fig. 10 and Fig. 11. They show a good convergence behavior.
Figure 10: Convergence of the tip-displacements for the quadrant cantilever: (a) $x$-displacement $u_{tip}^x$, (b) $y$-displacement $u_{tip}^y$, (c) $z$-displacement $u_{tip}^z$, (d) rotation $\phi_{tip}$.
Figure 11: Convergence of inner forces for the quadrant cantilever at point A: (a) normal force $\tilde{N}$, (b) bending moment $\tilde{M}_n$, (c) bending moment $\tilde{M}_b$, (d) torsional moment $\tilde{M}_T$.

There is no analytical solution of this problem available for assessment of accuracy. Still the results can be compared to the Bernoulli beam (BB) formulation as proposed and verified in [19]. Since it is an edge beam of a surface with open knot vector, the control points of the center line of BB can easily be coupled to the matching control points of the surface, when refining correspondingly. Consequently, a reference solution can be obtained by coupling the two independent structural elements – Kirchhoff-Love shell and Bernoulli beam – in a strong sense. This is equivalent to the embedded hinge-attached beam. The relative error $\mu$ can be computed by Eq. (39). Note that the result of the embedded beam is compared to result of the correspondingly refined BB set-up in order to separate embedding and discretization error. The discretization error was already proven in the respective papers [7, 19]. Fig. 12 shows the relative error $\mu$ plotted against the number of elements for different polynomial degrees.

$$\mu = \frac{\|O_{BB} - O_{POB}\|}{\|O_{BB}\|}$$  (39)
Figure 12: Comparison of the tip-displacements of the plug-on beam with the Bernoulli beam for the quadrant cantilever: (a) $x$-displacement $u^{\text{tip}}_x$, (b) $y$-displacement $u^{\text{tip}}_y$, (c) $z$-displacement $u^{\text{tip}}_z$, (d) rotation $\phi^{\text{tip}}$

The implementation shows good convergence to the results of the BB and therefore also to the correct solution. Note that the definition of the rotation angle is different from the BB formulation. Consequently, this DOF is actually not comparable. Therefore, another error norm is used for the convergence plot of the rotation in Fig. 12(d). The results are compared to an extensively refined patch with embedded beam (POB2048):

$$\mu_\phi = \frac{\|\text{POB2048} - \text{POB}\|}{\|\text{POB2048}\|} \quad (40)$$

4.1.2. Mainspring

The main-spring example provides a pure bending example, which is highly nonlinear. It is chosen in order to verify the beam formulation in a configuration, where the beam is not corresponding to the edge of the NURBS patch, i.e. cannot be modeled by a separate edge beam. The cantilever is modeled by a Kirchhoff-Love shell [7]. Two beams are plugged onto the trimming lines along the two sides (see Fig. 13). Respective bending moments are applied at the tip of each element corresponding to $M = \frac{4p}{L^2}EI$ in order to bend a beam to a circle. The mesh is shown in Fig. 13 and a polynomial degree of $p = 3$ is used. Fig. 14(a) shows the deformed geometry for several load steps, including the parameter lines of the NURBS patch which clearly demonstrate the simulation of a trimmed entity.
Figure 13: Initial geometry of the mainspring with input parameters and trimmed mesh with initial control point net. The virtual volume of the beam is depicted in blue.

The reference solution of the displacements can be determined by the respective circular segment.

\[
\begin{align*}
    u_{\text{ref}} &= L - \sin \left( \frac{L}{R} \right) \cdot R, \\
    v_{\text{ref}} &= \left( 1 - \cos \left( \frac{L}{R} \right) \right) \cdot R
\end{align*}
\]  

(41)

Figure 14: Computed results of the mainspring: (a) Deformed geometry for ten load steps, (b) Numerical results compared to the analytic solution.

4.1.3. Formfinding of a tent

The main purpose of this example is to test the independent rotation of the beam, since the example in Section 4.1.1 did not provide a proper possibility to compare the rotation angle. In this section, a formfinding example with a membrane supported by a circular beam and edge cables is chosen as shown in Fig. 15. The membrane and cable elements are defined by a prestress. The beam has elastic properties and serves as a soft support for the membrane in the formfinding process. The membrane is modeled as single patch with a \( C^0 \)-continuity in the middle and the beam is placed at this middle knot of the parameter space. The isogeometric membrane formulation of Philipp et al. [15] is used. Here, the initial correction angle for the cross section \( \Psi \) is applied to align \( B_3 \) to the \( y \)-axis. The DOF \( \psi \) aligns
the deformed configuration to the $y$-axis, since this set-up is symmetric and the beam should not twist (cf. Fig. 15(b)). The surface has up to 104 elements with $p = 4$ in beam direction and up to 100 elements with $p = 3$ in the other direction.

![Diagram](image)

Figure 15: Formfinding of a tent supported by a circular beam and edge cables. The blue colored beams are only visualized, but not part of the CAD modeling: (a) perspective view of undeformed and deformed configuration, (b) side view with applied correction angles of the cross section in undeformed and deformed configuration.

The displacement field can be compared to a tent, which is modeled by two strongly coupled NURBS patches and a BB beam. The displacements are the same for the embedded beam and the coupled BB with two patches. No difference is visible to the naked eye in Fig. 16.

![Displacement](image)

Figure 16: Displacement in z-direction: (a) Bernoulli Beam [19] (b) Plug-on beam

As mentioned above, the beam should not twist due to the symmetry and therefore not exhibit a torsional moment. Fig. 17 shows the mean of the absolute torsional moment $\|M_T\|$ over the whole beam. The error decreases with increasing number of elements.
4.2. Application examples

Further examples are shown in order to give an impression of the potential of such embedded curve plug-ons.

4.2.1. Edge-independent Line Loads And Supports

Curve elements can also concern boundary conditions. This happens e.g. for a forming process. A simplified simulation of this process can be done by blocking all displacement of a curve, which corresponds to the edge of the forming body, by a line support. In order to form the sheet a pressure line load is applied with offset to the supports. The set-up is shown in Fig. 18(a). Note that the lower body is only visualized for illustration reasons in order to show the potential of the formulation. The resulting deformation with von Mises stress plot is depicted in Fig. 18(b). It can be seen that the stress distribution is independent of the element borders. Good results can already be gained with a comparatively coarse mesh.

![Figure 18: (a) Set-up of the forming example with support and loads, (b) deformed shell with von Mises stresses. The upper shell is computed, the lower body is only depicted for illustration reasons.](image)

4.2.2. Mass manipulation

The embedded curve elements can also be attributed with mass. This enhancement can be used e.g. for eigenvalue or dynamic analyses. In the following, an eigenvalue determination of a simply supported plate is shown. The aspect ratio of this plate is 2 : 1. It can be modeled with a quite coarse discretization and the additional mass is independent of the knot lines. In the following, the added curve element is called beam even though it is without proper stiffness (i.e. a nonstructural mass). This setup has been selected in order to be able to separate effects of additional stiffness.
and mass in the results. In consequence, the beam element has “only” a virtual cross section area of $A = 1.0$. The density of the curve element is increased from zero, i.e. no additional mass is applied, up to a ratio of 10 between the mass per length of plate and beam. Obviously a combination of stiffness and mass is also possible.

Plate: 
- $t = 0.01$ m
- $\rho = 1.0$ kg/m$^3$
- $\bar{\rho} = 0.01$ kg/m

Beam: 
- $\rho = \rho_{\text{beam}}$ [kg/m]
- $A = 1.0$ m$^2$
- $\bar{\rho} = \rho_{\text{beam}} \cdot A$ [kg/m]
- $EI = 0.0$ kNm$^2$

No additional stiffness is applied. The blue volume only visualizes the additional mass.

This example shows that one can manipulate the eigenvalue problem by one-dimensional added masses which can be located independently of any parameter line (see Fig. 20). The eigenfrequency decreases with increasing density of the curve element, since the whole system becomes heavier, which increases the inertial forces. Furthermore, the eigenforms are distorted by the additional mass and even the order of the eigenforms can be changed cf. Fig. 20(b) and Fig. 20(c).

![Figure 19: Set-up of a mass manipulated, simply supported plate with mesh. The additional weight at the curve is varied by the respective density. No additional stiffness is applied. The blue volume only visualizes the additional mass.](image)

![Figure 20: First four eigenforms of the mass manipulated plate (Fig. 19) with respective eigenangular frequency $f$ for: (a) $\rho_{\text{beam}} = 0.00$, (b) $\rho_{\text{beam}} = 0.05$ and (c) $\rho_{\text{beam}} = 0.10$](image)
4.2.3. Ribbing of a plate

A further application is the ribbing of a plate respectively shell by beams. A square plate with a circular ribbing in the middle and straight ribbing from the corners to the circle is chosen as qualitative example. The surface is modeled by a single patch and is loaded by a surface dead load (Fig. 21(a)). The ribs become apparent in the deformed configuration (Fig. 21(b)) in a geometrically nonlinear analysis.

![Figure 21: A hinge supported plate with ribbings modeled by embedded beams: (a) Set-up and (b) deformed system](image)

5. Conclusions and outlook

This contribution proposes a concept for modification and combination of isogeometric models by using a nested parametrization. The nested parametrization allows a strong form of coupling, since the nested element is expressed with the control points of the super element. The super element can be thus subjected to local effects independent of its parametrization.

In this paper, the general concept was first illustrated by the geometry description with the example of a curve embedded into a surface. The geometric description of the local continuum can be used as basis for the formulations in governing equations of the statement of the problem, such as the equilibrium equations in structural mechanics. The application of the approach is demonstrated for two embedded nonlinear beam formulations together with related extensions like mass, supports and loads. The plug-ons were implemented in the structural analysis code at the authors’ Chair. Verification was done by comparing the results to analytic solutions and other formulations respectively discretizations. Furthermore, some fields of application were revealed.

Further potential of the concept can be seen in the application of nonzero initial displacements, which are not bounded to control points. Most notably, forming processes can be simulated if appropriate material models are available. Also other types of loads, e.g. thermal loads, can be applied. Furthermore, it could be useful for contact formulations, since a mesh independent description of the contact area becomes possible. If the control points in the parameter space of the super element are released, a movement of the curve inside the patch is possible. Also surfaces inside the surface can be described like this, which would allow e.g. local reinforcements by additionally welded sheets. The concept may also be transferred to other fields than structural mechanics.

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6. References


